Int. J. Heat Mass Transfer. Vol. 10, pp. 1509–1520. Pergamon Press Ltd. 1967. Printed in Great Britain

A GENERALIZED GALERKIN–KANTOROVICH TREATMENT OF TRANSIENT EVAPORATION THROUGH A FINITE REGION

HOWARD E. BETHEL[*]

Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

(*Received* 5 *November* 1966 *and in revised form* 1 *May* 1967)

Abstract—This paper presents an approximate solution of transient evaporation through a finite region. A generalized Galerkin–Kantorovich method is employed to obtain the approximate solutions. Exactness and convergence properties of the approximate solutions are established by comparison with exact solutions for two special cases. The approximate solutions appear to be converging to the exact as the order of approximation is increased. Furthermore, accurate solutions are obtained for a computationally feasible order of approximation (the fifth). The physical aspects of the approximate solutions are presented and discussed.

NOMENCLATURE

$a_j(\tau)$,	undetermined coefficients in approximating function u_N;
c,	molar density of solution;
i, j,	indices usually ranging over 1, 2, . . . , N;
t,	time;
$u(\tau, \eta)$,	nondimensionalized concentration of vapor, equation (5);
$u_N(\tau, \eta)$,	approximating function for variable $u(\tau, \eta)$, equation (17);
u_k,	collocation points, equation (22);
$u_{\max}$,	maximum value of $u(\tau_0, \eta)$ at a particular instant of time τ_0;
y,	coordinate normal to interface;
D_{AB},	binary diffusivity for system A–B;
G,	molar flux;
H_i,	set of weighting functions;
L,	reference distance;
N,	number of unknown coefficients in the approximating function $u_N(\tau, \eta)$;
R_N,	residual, equation (18);
T,	value of τ at onset of steady state, that is, earliest time for which $a_j(\tau)$ differ from a_{js} by less than 10^{-6};
V,	nondimensionalized flux, $GL/X_{A0}cD_{AB}$;
$X(\tau, y)$,	mole fraction;
X_{A0},	equilibrium gas-phase concentration;
$Y(n)$,	definite integral in system of integral conditions, equation (20).

Greek symbols

α_0,	interface parameter, equation (7);
η,	nondimentionalized coordinate normal to interface, equation (5);
τ,	nondimensionalized time, equation (5).

Subscripts

s,	variable evaluated for the steady state;
A,	variable evaluated for vapor A;
B,	variable evaluated for gas B.

Superscript

$\cdot$,	derivative with respect to τ.

---

[*] Captain, USAF and Research Engineer, Hypersonic Research Laboratory.

1. INTRODUCTION

The transient evaporation of a liquid A through a semi-infinite region filled with gas B was solved by Arnold [1] by a similarity transformation which reduced the governing partial differential equation to an ordinary differential equation. The problem considered in this paper is the transient evaporation of a liquid A through a finite region filled with gas B. Thus, in this problem the concentration of A is maintained as zero at some finite distance, L, above the surface of liquid A. For times $t < 0$, liquid A is not exposed to gas B. At the instant of time $t = 0$, the interface between liquid A and gas B is established. The concentration of A at the interface $y = 0$ is then X_{A0}, the equilibrium gas-phase concentration. The physical model is shown schematically in Fig. 1.

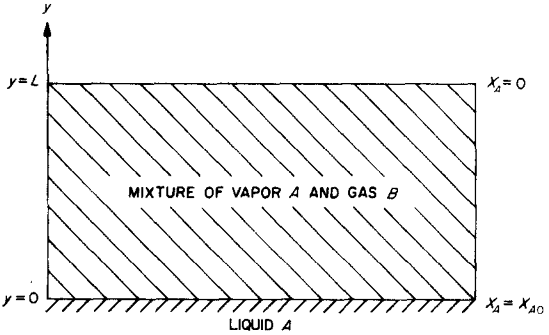


Fig. 1. Transient evaporation across a finite region.

The level of the liquid A is maintained at position $y = 0$ at all times. The entire system is maintained at a constant temperature and pressure. The vapor A and gas B are assumed to form an ideal gas mixture. It is further assumed that B is insoluble in A. The governing equation for this evaporation process under these assumptions is as follows [2]:

$$\frac{\partial X_A}{\partial t} = D_{AB}\frac{\partial^2 X_A}{\partial y^2} + \frac{D_{AB}}{1 - X_{A0}}\left.\frac{\partial X_A}{\partial y}\right|_{y=0}\frac{\partial X_A}{\partial y} \qquad (1)$$

where

X_A, mole fraction of vapor A;
y, normal distance above liquid A;
t, time;
D_{AB}, binary diffusivity for system A–B, taken as a constant;
X_{A0}, equilibrium gas-phase concentration.

The boundary conditions are

$$y = 0 \qquad X_A = X_{A0} \qquad (2)$$

$$y = L \qquad X_A = 0 \qquad (3)$$

and the initial condition is

$$t = 0 \qquad X_A = 0. \qquad (4)$$

It is convenient to rewrite the above problem in terms of the nondimensional variables

$$u = \frac{X_A}{X_{A0}}, \quad \eta = \frac{y}{L} \quad \text{and} \quad \tau = \frac{tD_{AB}}{L^2}. \qquad (5)$$

The governing differential equation written in nondimensional form is given by

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} + \alpha_0\left.\frac{\partial u}{\partial \eta}\right|_{\eta=0}\frac{\partial u}{\partial \eta} \qquad (6)$$

where

$$\alpha_0 = \frac{X_{A0}}{1 - X_{A0}}. \qquad (7)$$

The boundary conditions are

$$\eta = 0 \qquad u = 1 \qquad (8)$$

$$\eta = 1 \qquad u = 0 \qquad (9)$$

and the initial condition is

$$\tau = 0 \qquad u = 0. \qquad (10)$$

For the case $\alpha_0 = 0$, equation (6) reduces to the diffusion equation and the resulting problem can be readily solved [3, 4]. Also, the steady state solution of (6) can be obtained for all values of α_0. The purpose of this paper is to present approximate unsteady state solutions for nonzero values of α_0. The solutions are obtained with a generalized Galerkin–Kantorovich method which is outlined in the next section.

2. THE GENERALIZED GALERKIN–KANTOROVICH METHOD

The generalized Galerkin–Kantorovich method can be briefly summarized as follows [5, 6, 7]. Let the given partial differential equation be denoted in operator notation as $F[w(\zeta,\xi)] = 0$ with suitable boundary conditions on $w(\zeta,\xi)$. An approximate solution is taken in the form

$$w(\zeta,\xi) \approx w_N(\zeta,\xi) = \phi_0(\zeta,\xi) + \sum_{j=1}^{N} a_j(\zeta)\,\phi_j(\zeta,\xi) \tag{11}$$

where $\phi_0(\zeta,\xi)$ satisfies the nonhomogeneous boundary conditions and the functions $\phi_j(\zeta,\xi)$ satisfy homogeneous boundary conditions for all values of the parameters $a_j(\zeta)$. Further, the functions $\phi_j(\zeta,\xi)$ are taken to be the first N members of a complete set of functions, thus, in the limit of large values of N, it is possible to find coefficients a_j such that the approximating function $w_N(\zeta,\xi)$ is equal to the function $w(\zeta,\xi)$. For finite values of N, $F[w_N(\zeta,\xi)] = R_N \neq 0$ where R_N is called the residual and in some sense reflects the degree of closeness of w_N to w.

A system of integral conditions for the determination of the coefficients a_j is generated by requiring that the residual be orthogonal to a set of weighting functions

$$\int_{\xi_1}^{\xi_2} H_i(\xi)\,R_N\,\mathrm{d}\xi = \int_{\xi_1}^{\xi_2} H_i(\xi)\,F[w_N(\zeta,\xi)]\,\mathrm{d}\xi = 0 \tag{12}$$

$$i = 1, 2, \ldots, N.$$

The weighting functions $H_i(\xi)$ are taken as any set of linearly independent functions. The set of N linearly independent integral conditions represented by equation (12) will be ordinary differential equations for the $a_j(\zeta)$'s and will be nonlinear when the operator $F[w(\zeta,\xi)]$ is nonlinear.

3. FORMULATION OF THE APPROXIMATE SOLUTION

In this section, an Nth order solution will be formulated in the spirit of the formulations of [8]. The approximating function of the form (11) for the dependent variable $u(\tau,\eta)$ can be derived by writing

$$u_M(\tau,\eta) = \sum_{j=1}^{M} b_j(\tau)\,\eta^{j-1} \tag{13}$$

and requiring that this function satisfy the boundary conditions for all values of the b_j's. Satisfaction of (8) gives

$$b_1 = 1 \tag{14}$$

and satisfaction of (9) yields

$$b_2 = -\Big(1 + \sum_{j=3}^{M} b_j\Big). \tag{15}$$

Combining equations (14) and (15) with equation (13) gives

$$u_M(\tau,\eta) = 1 - \eta + \sum_{j=3}^{M} b_j[-\eta + \eta^{j-1}]. \tag{16}$$

The approximating function u_M can now be rewritten in the form (11) by adjusting the upper limit of the summation so that it reflects the number of remaining unknown functions, $M - 2$, which will be called N.

$$u_N(\tau,\eta) = 1 - \eta + \sum_{j=1}^{N} a_j(\tau)\,[-\eta + \eta^{j+1}]. \tag{17}$$

Equation (17) identically satisfies the boundary conditions (8) and (9) irrespective of the values of the a_j's.

If the approximating function (17) is employed in the governing equation (6), the expression for the residual R_N is found to be

$$\begin{aligned} R_N = {} & \sum_{j=1}^{N} \dot{a}_j[-\eta + \eta^{j+1}] - \sum_{j=1}^{N} a_j(j+1)(j)\eta^{j-1} \\ & + \alpha_0\Big[1 + \sum_{j=1}^{N} a_j\Big] \\ & \times \Big[-1 + \sum_{j=1}^{N} a_j\{-1 + (j+1)\eta^j\}\Big] \end{aligned} \tag{18}$$

where $\dot{a}_j = \mathrm{d}a_j/\mathrm{d}\tau$. When the residual is substituted into the orthogonality relation (12) and the set of weighting functions taken as

$H_i(\eta) = \eta^{i-1}$, the system of N integral conditions becomes

$$\sum_{j=1}^{N} \dot{a}_j[-Y(i+1) + Y(i+j+1)]$$

$$= \sum_{j=1}^{N} a_j(j+1)jY(j+i-1) - \alpha_0\left[1 + \sum_{j=1}^{N} a_j\right]$$

$$* \left[-Y(i) + \sum_{j=1}^{N} a_j\{-Y(i) + (j+1)\,Y(j+i)\}\right] \qquad (19)$$

$$i = 1, 2, \ldots, N$$

where

$$Y(n) = \int_0^1 x^{n-1}\,\mathrm{d}x = 1/n. \qquad (20)$$

This system of equations determines the N unknowns $a_j(\tau)$ $j = 1, 2, \ldots, N$. However, before this system of equations can be solved for the transient concentration profiles $u_N(\tau, \eta)$, initial values of the a_j's must be determined.

Initial values of the a_j's are determined by requiring that the approximating function $u_N(\tau, \eta)$ satisfy the initial condition $u(0, \eta) = 0$. In terms of the approximating function this condition requires that

$$u_N(0, \eta) = 1 - \eta + \sum_{j=1}^{N} a_j(0)\left[-\eta + \eta^{j+1}\right] = 0 \qquad (21)$$

where the $a_j(0)$'s are the initial values of the a_j's. For a finite number of $a_j(0)$'s it is not possible to identically satisfy equation (21). For example, in the first order of approximation there is available only one coefficient $a_1(0)$. However, from equation (21) one finds

$$a_1(0) = \frac{\eta - 1}{(-\eta + \eta^2)}$$

and since $a_1(0)$ is a constant it is only possible to satisfy (21) at a single point. Hence, the initial condition can only be satisfied in an approximate manner. The initial values of the coefficients can be determined by requiring that $u_N(0, \eta)$ be orthogonal to the weighting functions $H_i(\eta)$ as was done with the residual. However, a collocation procedure was found to give more exact initial conditions. The scheme employed to generate the initial values of the coefficients is to require that equation (21) be satisfied at the points $u_k = k/(N+1)$, $k = 1, 2, \ldots, N$. The resulting system of equations is given by

$$\sum_{j=1}^{N} a_j(0)\left[-\eta_k + \eta_k^{j+1}\right] = \eta_k - 1. \qquad (22)$$

This system of N algebraic equations can be solved for the N values $a_j(0)$.

We now have all the elements necessary to obtain the approximate solution. The solution procedure is to specify a desired value of α_0 and N and then to first obtain the initial values of the a_j's from equation (22). With these initial conditions the system of ordinary differential equations (19) can be solved by any standard finite difference method (a variable-step Adams–Moulton scheme was employed in this study). Once the $a_j(\tau)$ values are known at some particular instant of time the concentration profiles can be recovered from the equation for $u_N(\tau, \eta)$, equation (17).

The above formulation illustrates the coupled use of orthogonality and collocation methods. These methods are but two of a number of error distribution principles which can be used to obtain approximate solutions of differential equations [7]. The method of weighted residuals encompasses and unifies consideration of the above methods. Crandall [9] originated this unifying concept and a comprehensive review of it is given by Finlayson and Scriven [10].

4. EXACTNESS AND CONVERGENCE

The convergence of the generalized Galerkin–Kantorovich method when applied to the transient evaporation problem would be anticipated since the completeness of the set of functions ϕ_j can be inferred from the Weierstrass Approximation Theorem. The Weierstrass Approximation Theorem states:

If the function $v(x, y)$, $\partial v/\partial x$ and $\partial v/\partial y$ are continuous in a closed and bounded region D, then for any $\epsilon > 0$ there exists a polynomial $p(x, y)$ such that in D, $v - p < \epsilon$, $(\partial v/\partial x) - (\partial p/\partial x) < \epsilon$ and $(\partial v/\partial y) - (\partial p/\partial y) < \epsilon$ [11].

From the statement of the problem, equations (6) through (10), it is clear that for $\tau > 0$ the dependent variable u and its derivative $\partial u/\partial\eta$ are continuous in the closed and bounded region $0 \leqslant \eta \leqslant 1$. Thus, the dependence of u upon η can be approximated to any desired degree of accuracy by a sum of polynomials in η. That is, the approximating function is mathematically complete and it is possible to find coefficients b_j such that

$$\lim_{M\to\infty} u_M(\tau, \eta) = u(\tau, \eta).$$

The author is not aware of a general proof of the convergence of the generalized Galerkin–Kantorovich method which would apply to this problem. Consequently, the convergence and exactness properties of the approximate solutions can only be inferred by comparing these results with available exact solutions.

For the transient evaporation problem there are two special cases for which exact solutions exist. The first, and most useful, is for $\alpha_0 = 0$. In this case, equation (6) reduces to the diffusion equation and the exact solution has been obtained by classical techniques [3, 4]. The exact solution is given by

$$u(\tau, \eta) = 1 - \eta - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin(n\pi\eta)\exp(-n^2\pi^2\tau) \tag{23}$$

and can be evaluated to any desired degree of accuracy. The second case is the steady state. For the steady state $\partial u/\partial\tau = 0$ and equation (6) reduces to an ordinary differential equation which can be exactly integrated for all values of α_0

$$\begin{aligned} &\alpha_0 = 0 \qquad u_s(\eta) = 1 - \eta \\ &\alpha_0 \neq 0 \qquad u_s(\eta) = 1 - \frac{1}{\alpha_0} \\ &\qquad\qquad \times (\exp\{[\ln(\alpha_0 + 1)]\eta\} - 1). \end{aligned} \tag{24}$$

The degree to which the approximating function satisfies the initial conditions also affects the exactness of the approximate solutions.

To establish the convergence and exactness properties of the approximate solutions, we will first consider the limiting conditions, that is, the initial conditions and steady state solutions. The initial conditions are obtained from equation (22) and for a particular order of approximation are the same for all values of α_0. The initial conditions for the first eight orders of approximation are presented in Fig. 2. The approximate initial conditions tend uniformly toward the exact initial condition with increasing N. However, the initial conditions are considerably in error for small values of η due to the nature of the initial condition, $u(0, \eta) = 0$, and boundary condition at $\eta = 0$, $u(\tau, 0) = 1$. This error is rapidly diminished as the solution proceeds in time as shall be seen subsequently. The steady state solutions of the generalized Galerkin–Kantorovich method are obtained

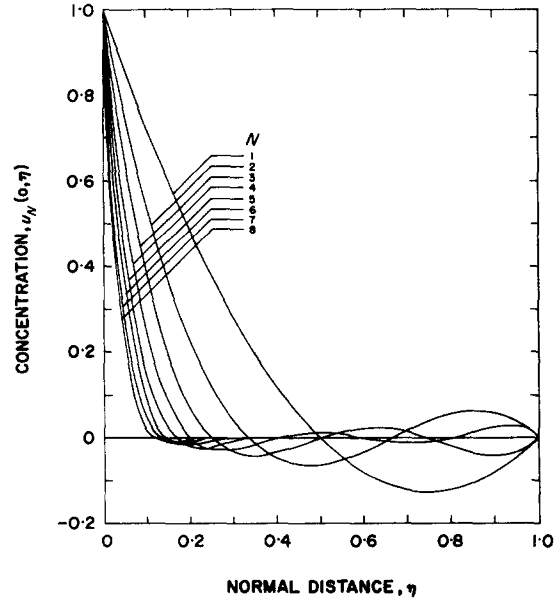


FIG. 2. Initial concentration profiles.

Table 1. Steady state solutions for $\alpha_0 = 1\cdot0$

η	Nondimensional concentration, $u_N(\tau_s, \eta)$					
	Order of approximation, N					Exact
	1	2	3	4	5	
0	1·0	1·0	1·0	1·0	1·0	1·0
0·0500	0·96583	0·96475	0·96473	0·96474	0·96474	0·96474
0·1000	0·93000	0·92832	0·92822	0·92823	0·92823	0·92823
0·1500	0·89250	0·89063	0·89042	0·89043	0·89043	0·89043
0·2000	0·85333	0·85162	0·85129	0·85130	0·85130	0·85130
0·2500	0·81250	0·81125	0·81077	0·81079	0·81079	0·81079
0·3000	0·77000	0·76944	0·76884	0·76886	0·76886	0·76886
0·3500	0·72583	0·72614	0·72542	0·72544	0·72544	0·72544
0·4000	0·68000	0·68128	0·68048	0·68049	0·68049	0·68049
0·4500	0·63250	0·63481	0·63395	0·63396	0·63396	0·63396
0·5000	0·58333	0·58667	0·58578	0·58579	0·58579	0·58579
0·5500	0·53250	0·53680	0·53592	0·53591	0·53591	0·53591
0·6000	0·48000	0·48513	0·48429	0·48428	0·48428	0·48428
0·6500	0·42583	0·43160	0·43084	0·43083	0·43083	0·43083
0·7000	0·37000	0·37617	0·37551	0·37550	0·37550	0·37550
0·7500	0·31250	0·31876	0·31822	0·31821	0·31821	0·31821
0·8000	0·25333	0·25932	0·25891	0·25890	0·25890	0·25890
0·8500	0·19250	0·19778	0·19751	0·19750	0·19750	0·19750
0·9000	0·13000	0·13409	0·13394	0·13393	0·13393	0·13393
0·9500	0·06583	0·06818	0·06813	0·06813	0·06813	0·06813
1·0000	0	0	0	0	0	0

by solving the system of algebraic equations which results from setting $\dot{a}_j = 0$ in equation (19). The approximate steady state concentration profiles for $\alpha_0 = 1$ are compared with the exact profiles in Table 1. The approximate profile is identical to the exact for orders of approximation $N \geqslant 4$. Essentially, the same conclusion is valid for other values of α_0.

As a consequence of examining the behavior of the limiting conditions, it is clear that initially there is some error in the approximate solution but that this error vanishes as τ becomes large (for sufficiently high orders of approximation). The exactness of the approximate solutions for intermediate values of τ can be determined for the special case $\alpha_0 = 0$. The exactness of the approximate solutions when $\alpha_0 \neq 0$ can only be inferred from the results for $\alpha_0 = 0$. The percent differences between the exact and approximate solutions $[(u_N - u)/u_{\max}] * 100$ for the case $\alpha_0 = 0$, $N = 5$ and $\eta = 0\cdot1$, $0\cdot5$ and $0\cdot9$ are tabulated in Table 2.

Table 2. Percent differences for fifth order solution with $\alpha_0 = 0$

τ	Percent difference, $[(u_N - u)/u_{\max}]$ 100		
	$\eta = 0\cdot1$	$\eta = 0\cdot5$	$\eta = 0\cdot9$
0·02	1·40302	3·80826	−0·27607
0·04	0·42074	1·20893	+0·14632
0·06	0·26117	0·68315	0·13964
0·08	0·18833	0·51090	0·12533
0·10	0·14334	0·41134	0·11081
0·12	0·11225	0·33636	0·09569
0·14	0·08944	0·27590	0·08111
0·16	0·07206	0·22645	0·06792
0·18	0·05847	0·18588	0·04739
0·20	0·04766	0·15258	0·04666
0·25	0·02887	0·09315	0·02871
0·50	0·00244	0·00790	0·00244
0·75	0·00021	0·00067	0·00021
1·00	0·00002	0·00006	0·00002
1·25	0·00000	0·00000	0·00000

The differences are greatest for small values of τ and decrease to zero as τ becomes large. The percent differences given in Table 2 indicate the errors which might be expected for any particular intermediate value of τ. The maximum error is less than 4 per cent and the error is less than 1 per cent for $\tau \geqslant 0{\cdot}06$.

The convergence of the limiting conditions of the approximate solutions was shown to be of a uniform nature. The convergence of the approximate solutions for intermediate values of τ can be inferred by examining the properties of these solutions as a function of order of approximation for some fixed value of τ. The per cent differences for $\tau = 0{\cdot}1$ and $\alpha_0 = 0$ are given in Fig. 3. The approximate solutions clearly converge toward the exact solutions.

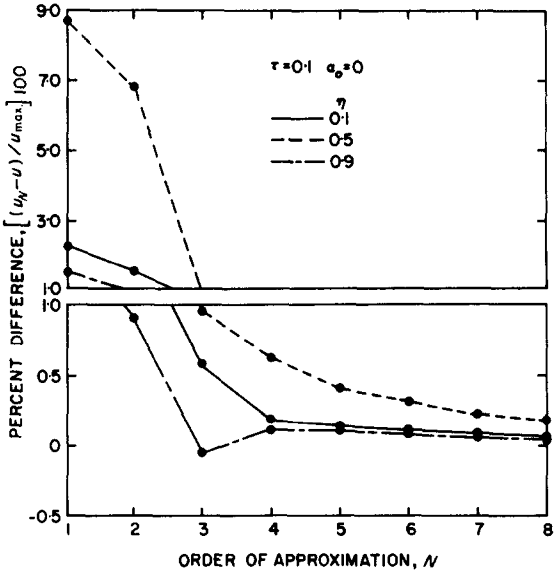
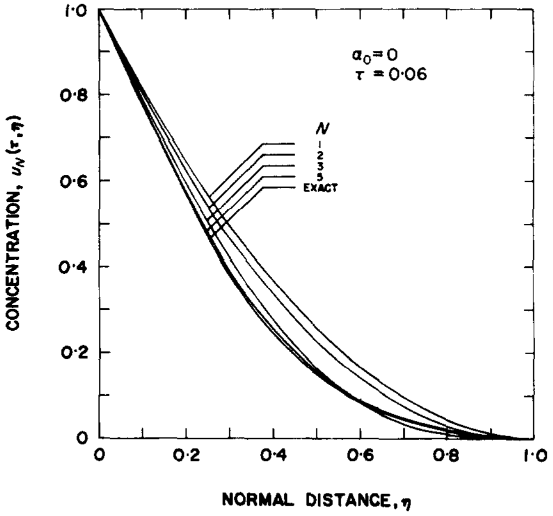
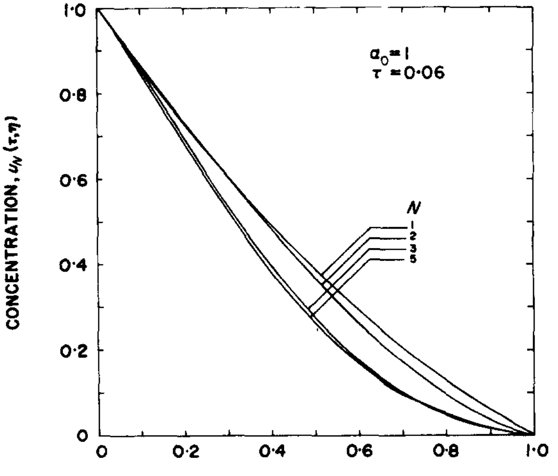


FIG. 3. Percent differences as a function of order of approximation.

The concentration profiles as a function of order of approximation for $\tau = 0{\cdot}06$ and $\alpha_0 = 0$ are presented in Fig. 4a. Only the curves for the first three orders and the fifth order are plotted in the figure. The fourth-order curve falls about a point above the fifth-order curve and the sixth through eighth order solutions are between the fifth order curve and the exact solution. The results for each successive order of approximation are closer to the exact than previous orders of approximation. The approximate solutions for nonzero values of α_0 are similar to those for $\alpha_0 = 0$. The concentration profiles as a function of order of approximation for $\tau = 0{\cdot}06$ and $\alpha_0 = 1$ are presented in Fig. 4b. Again, only the curves for the first three orders and the fifth order are presented. The fourth-order solution is about a point above the fifth order curve. The sixth and seventh order solutions are a point or so below



(a) $\alpha_0 = 0$



(b) $\alpha_0 = 1{\cdot}0$

FIG. 4. Concentration profiles as a function of order of approximation.

the fifth-order curve. One characteristic noted for solutions with nonzero α_0 is apparent in Fig. 4b. For a small range of η near $\eta = 0$ the first-order solution is more exact than the second. However, in an overall sense, the second order solution is more exact than the first. This characteristic was only observed for values of $\tau < 0{\cdot}1$.

An important facet of any approximate method is the determination of a particular order of approximation which is a good compromise between the requirements for accuracy and computational feasibility. For this problem the increase in accuracy in going to values of $N > 5$ does not seem to be justified in view of the increased computation time necessary to do so. For instance, the results presented in Fig. 3 for the fifth order of approximation are less than 0·5 per cent in error and those for the eighth order are less than 0·2 per cent in error. However, the computation time for the fifth order is about 25 s whereas that for the eighth order is over 250 s (duration in execution on IBM 7094). Consequently, in the following section results for the fifth order of approximation are presented.

5. PRESENTATION AND DISCUSSION OF RESULTS

In the previous section, we have presented the approximate solution technique and established the convergence and exactness properties of the approximate solutions. In this section, the physical aspects of these solutions are analyzed. The phenomena which we are considering is diffusion controlled evaporation across a finite region. A liquid A evaporates into a region originally filled entirely with gas B. The creation of the interface between the liquid and gas at time zero produces a concentration gradient which causes a diffusion of vapor and a counter diffusion of gas. Bulk motion of the vapor and gas also occurs because of the displacement effect necessary to maintain constant pressure. Fluxes result from the diffusion and bulk motion. A flux at one instant of time causes a modified concentration distribution at a later instant which in turn gives rise to a new flux. This transient process continues until a steady state is reached. In the steady state, the flux of vapor is constant across the region and that of the gas is zero. The concentration, concentration gradient and flux are probably the most important quantities for an understanding of this phenomena. In the next paragraph we will establish the equations necessary to evaluate these quantities as a function of time and location.

The equation for the rate of flow, or flux, of vapor is given by [1] .

$$X_B V_A - X_A V_B = -\frac{\partial u}{\partial \eta}. \tag{25}$$

Thus, the flux of vapor is dependent on the flux of gas V_B as well as the mole fractions X_A and X_B. Maintenance of constant pressure in the system requires a constant upward rate of flow of vapor and gas together. That is, the sum of the two fluxes across any plane must equal the rate of vapor formation at $\eta = 0$

$$V_A + V_B = V_{A0} = -(1 + \alpha_0)\frac{\partial u}{\partial \eta}\bigg|_0 \tag{26}$$

since the flux of gas is zero at the interface and $X_A + X_B = 1$. The governing partial differential equation (6) is obtained by combining equations (25) and (26) with the conservation of vapor equation

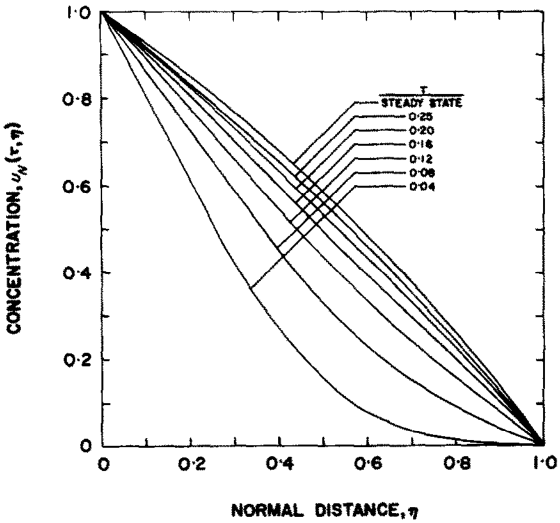
$$\frac{\partial u}{\partial \tau} = -\frac{\partial V_A}{\partial \eta}. \tag{27}$$

Integration of the governing equation yields the concentration distribution $u_N(\tau, \eta)$. This expression can be differentiated to give the concentration gradient $\partial u_N/\partial \eta$. An expression for the flux of vapor, V_A, is obtained by substituting equation (26) into equation (25). These quantities are for the vapor, however, corresponding expressions for the gas can be generated. The interface parameter α_0 appears in the above equations and is directly related to the equilibrium gas-phase concentration. The values of the interface parameter considered in this study are summarized in Table 3.

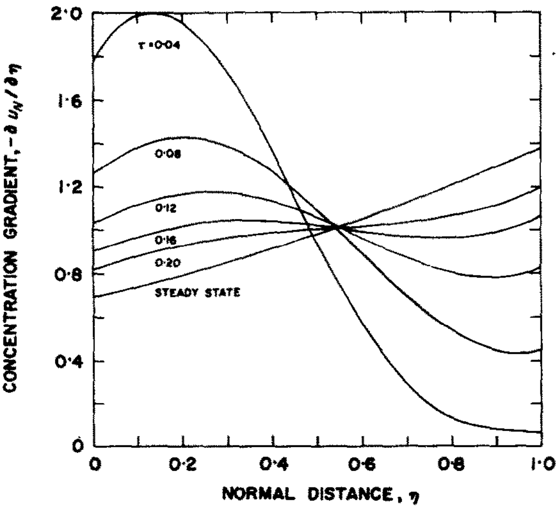
We will first examine the transient behavior of the approximate solutions for a particular value of α_0 and then indicate the effect of the interface parameter. The concentration, concentration gradient and flux as a function of time for $\alpha_0 = 1{\cdot}0$ are given in Fig. 5. Initially, the concentration is zero for all η except $\eta = 0$ where it is one. The concentration profiles develop rapidly toward the steady state profile. By the time $\tau = 0{\cdot}25$ the concentration distribution is close to the steady state. However, the approach is asymptotic in nature and it is not until $\tau = T = 1{\cdot}18194$ that steady state is reached. At time zero the concentration gradient and flux are zero for all η except $\eta = 0$ where they are infinite. These quantities develop as rapidly as the concentration profiles. In the steady state, the flux becomes constant across the region. The solutions for all values of α_0 evidence a behavior similar to that shown above for $\alpha_0 = 1{\cdot}0$. In the next paragraph, we will note the influence of the interface parameter on the solutions.

Table 3. Values of interface parameters

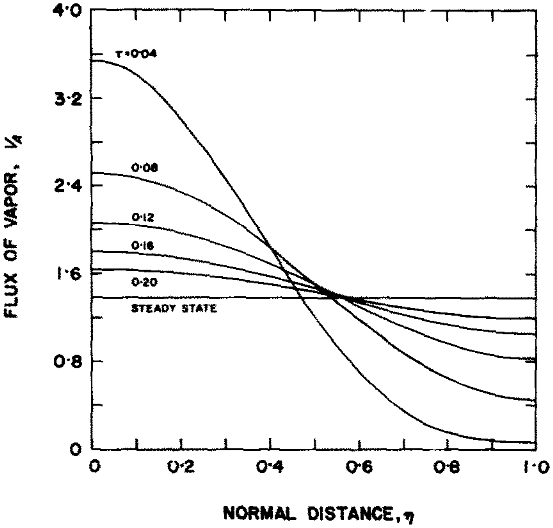
α_0	0	$\frac{1}{3}$	1	3	9
X_{A0}	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{9}{10}$



(a) Concentration



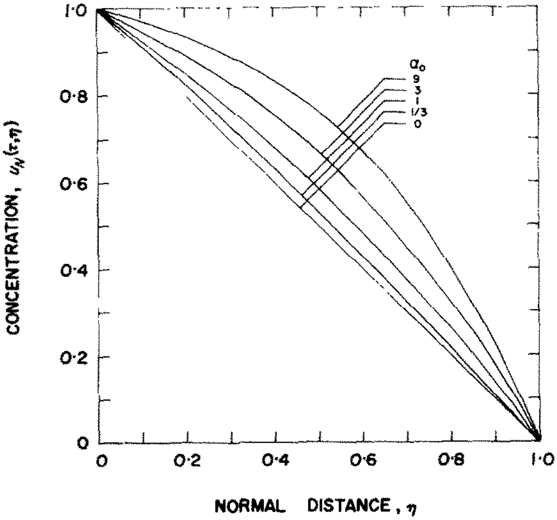
(b) Concentration gradient



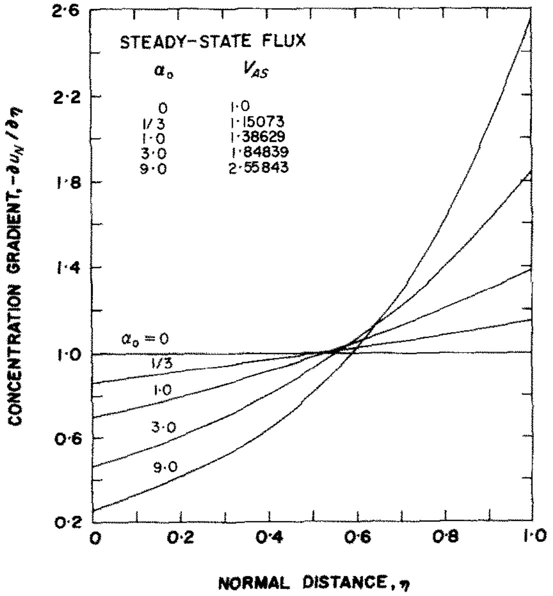
(c) Flux

FIG. 5. Transient behavior of approximation solution for $\alpha_0 = 1{\cdot}0$.

In Section 4, we observed that the initial condition is the same for all values of α_0. Consequently, the influence of the interface parameter on the approximate solutions is indicated by the steady state profiles which are presented in Fig. 6. As the interface parameter increases, the concentration profiles become fuller indicating a greater displacement of gas; the slope of the concentration gradient increases; and the steady state flux value increases. The interface parameter also influences the rate at which the steady state is approached. The vaporization rate as a function of time is given in Fig. 7. The steady state is approached more rapidly for larger values of α_0. Furthermore, the onset of steady state time, T, decreases sharply as α_0 increases.



(a) Concentration



(b) Concentration gradient

FIG. 6. Approximate solutions for steady state.

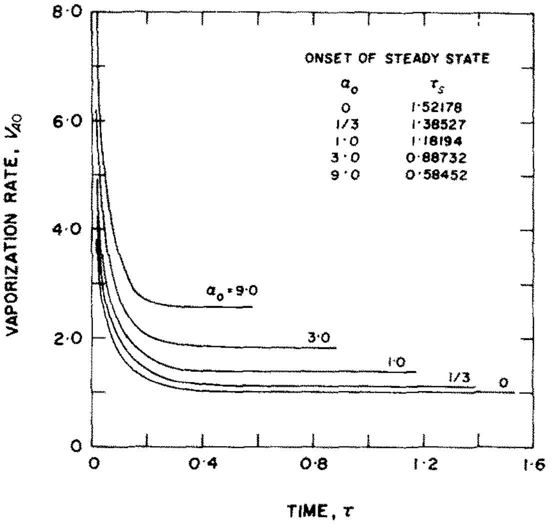


FIG. 7. Vaporization rate.

As a concluding point, let us consider the flux of vapor and the flux of gas once more. Expressions for these fluxes are derived from equations (25) and (26) and can be written

$$\left.\begin{aligned} V_A &= -\,u\alpha_0 \left.\frac{\partial u}{\partial \eta}\right|_0 - \frac{\partial u}{\partial \eta} \\ V_B &= -\,[(1+\alpha_0) - \alpha_0 u] \left.\frac{\partial u}{\partial \eta}\right|_0 + \frac{\partial u}{\partial \eta}. \end{aligned}\right\} \quad (28)$$

The fluxes are composed of a bulk motion term and a diffusion term. For the vapor these terms are additive. However, for the gas the diffusion opposes the bulk motion. The components of the fluxes at $\eta = 0·5$ for $\alpha_0 = 1·0$ are given in Fig. 8. While the total flux of gas becomes zero in the steady state, this does not mean that the gas is static. The component of

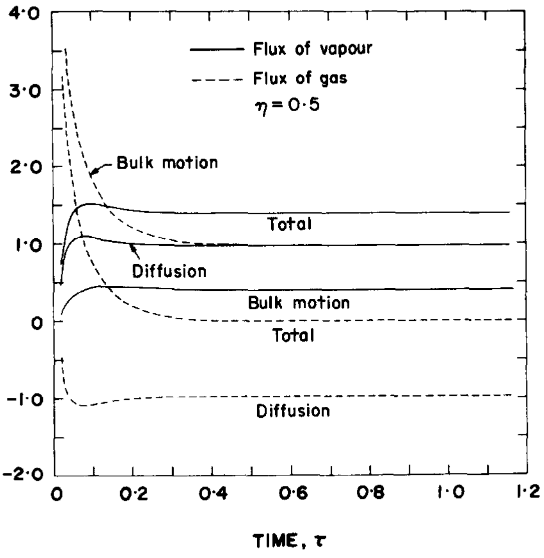


FIG. 8. Components of flux.

the flux of gas due to diffusion is equal in magnitude but opposite in direction to the component due to the bulk motion. This illustrates the dynamic nature of the steady state process. The existence of a steady state for diffusion controlled evaporation across a finite region is a significant difference from the process in an infinite region which does not admit a steady state.

6. CONCLUSION

In conclusion, the generalized Galerkin–Kantorovich method has been applied to the transient diffusion controlled evaporation across a finite region. The convergence and exactness properties of the approximate solutions were established by comparison with available exact solutions. The approximate solutions tend to converge as the order of approximation is increased. The exactness of the approximate solutions increases with increasing time and for the fifth order is less than one per cent for $\tau \geqslant 0{\cdot}06$. The fifth order of approximation was suggested as the best compromise between the requirements for accuracy and computational feasibility. The physical aspects of the solutions were presented and discussed. The most significant difference from solutions for the infinite region is the occurrence of a steady state for the finite region process.

The generalized Galerkin–Kantorovich method would appear to have many further applications in solving nonsimilar heat- and mass-transfer problems.

ACKNOWLEDGEMENTS

The author is indebted to Professor P. W. McFadden of Purdue University for identifying this as a problem which needed further study and to Professor D. E. Abbott of Purdue for motivating him to solve the problem. The author wishes to thank Dr. W. L. Hankey of the Aerospace Research Laboratories for his kind interest and stimulating discussion. Computational support was provided by the Digital Computation Division, Systems Engineering Group, Wright–Patterson Air Force Base.

REFERENCES

1. J. H. ARNOLD, Studies in diffusion. III. Unsteady-state vaporization and absorption, *Trans. Am. Inst. Chem. Engrs* **40**, 361 (1944).
2. R. B. BIRD, W. E. STEWART and E. N. LIGHTFOOT, *Transport Phenomena.* Wiley, New York (1960).
3. J. CRANK, *The Mathematics of Diffusion.* Clarendon Press, Oxford (1956).
4. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids*, 2nd edn. Clarendon Press, Oxford (1959).
5. L. V. KANTOROVICH, Use of the idea of the Galerkin method in the method of reduction to ordinary differential equations, *Prikl. Mat. Mekh.* **4**, 31 (1942).
6. L. V. KANTOROVICH and V. I. KRYLOV; *Approximate Methods of Higher Analysis.* Interscience, New York (1958).
7. L. COLLATZ, *The Numerical Treatment of Differential Equations*, 3rd edn. Springer-Verlag, Berlin (1960).
8. H. E. BETHEL, On the convergence and exactness of solutions of the laminar boundary-layer equations using the N-parameter integral formulation of Galerkin–Kantorovich–Dorodnitsyn, Ph.D. Dissertation, Purdue University (1966). Also ARL 66-0090.
9. S. H. CRANDALL, *Engineering Analysis.* McGraw-Hill, New York (1956).
10. B. A. FINLAYSON and L. E. SCRIVEN, The method of weighted residuals—a review, *Appl. Mech. Rev.* **19**, 735 (1966).
11. W. F. AMES, *Nonlinear Partial Differential Equations in Engineering.* Academic Press, New York (1965).

Résumé—Une solution approchée de l'évaporation transitoire à travers une région finie est présentée ici. On emploie une méthode généralisée de Galerkin–Kantorovich pour obtenir les solutions approchées. Les properiétés d'exactitude et de convergence des solutions approchées sont établies en les comparant avec des solutions exactes dans deux cas spéciaux. Il semble que les solutions approchées convergent vers les solutions exactes lorsqu'on augmente l'ordre d'approximation. De plus, des solutions précises sont obtenues avec un ordre d'approximation (le ciniquième) réalisable avec un calculateur. Les aspects physiques des solutions approchées sont présentés et discutés.

Zusammenfassung—Es wird eine Näherungslösung für die instationäre Verdampfung durch einen endlichen Bereich gegeben. Eine verallgemeinerte Galerkin–Kontorovich Methode wird angewandt, um die Näherungslösung zu erhalten. Genauigkeit und Konvergenz der Näherungslösungen wurden durch einen Vergleich mit den exakten Lösungen zweier Spezialfälle gewährleistet. Die Näherungslösungen scheinen mit erhöhter Näherungsordnung gegen den exakten Wert zu konvergieren. Ausserdem wurden genzue Lösungen erhalten für eine rechnerisch vertretbare Näherungsordnung (fünfte Ordnung). Die physikalischen Gesichtspunkte der Näherungslösungen wurden dargelegt und diskutiert.

Аннотация—В данной статье приводится приближенное решение для случая испарения в переходном режиме в конечном области. Обобщенный метод Галеркина–Канторовича используется для получения прилиженных решений. Точность и сходимость приближенных решений определяется сравнением с точными решениями для двух особых случаев. Оказывается, что приближенные решения стремятся к точным по мере увеличеня порядка аппроксимации. Довольно точные результаты получены в пятом приближении. Приводятся и обсуждаются физические аспекты приближенных решений.